Elliptic representation of the Boltzmann equation with validity for all degrees of anisotropy

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(Received 7 August 1998)

It is shown that by choosing an ellipsoid of revolution to describe the angular dependence of the velocity distribution function, the Boltzmann equation can be reduced to a set of two equations that have validity over a wide range of conditions. These equations reduce to the common two-term spherical harmonic expansion for nearly isotropic cases, but also properly describe highly anisotropic conditions. An example is given of the application of this approach to the Townsend discharge in helium over a very wide range of E/N.
[S1063-651X(99)03704-6]

PACS number(s): 51.10.+y, 51.50.+v

I. INTRODUCTION

The popular two-term spherical harmonic expansion of the Boltzmann equation [1–3] is very useful, but suffers from some serious drawbacks as the magnitude of the asymmetric component ($\vec{f}_1$) grows relative to the symmetric component (f_0). In fact, once the magnitude of $\vec{f}_1$ exceeds that of f_0, the distribution function becomes negative at some angles, and is thus nonphysical. Highly distorted distribution functions cannot be properly represented by this method. Furthermore, there is no mechanism in the resultant angular moment equations for limiting the disparity of these quantities.

Example exist of higher-order implementations of this type of expansion. For example, sixth-order solutions for some cases in N_2 have been obtained by Pitchford and Phelps [4] and again by Phelps and Pitchford [5] in which comparisons are made with a two-term expansion. They found that, at higher energies and fields, the inclusion of more terms made a significant difference. Similarly, Loffhagen and Winkler [6] performed time-dependent calculations for neon using as many eight terms, and found deficiencies with the two-term approximation in the early stages of relaxation.

At large anisotropy, beam formation has been studied by examining the Boltzmann equation at small angles with respect to the field axis. Riemann [7] gives such an example to first order in $1-\mu$, where μ is the direction cosine with the field axis. Long [8] performs a similar analysis for the special case of constant, isotropic cross sections, and compares this result with multiterm expansions of various order. Pitchford and Phelps [4] also found this approach to compare well with their six-term solutions for cases of large anisotropy.

Although the higher-order spherical harmonic expansions have been studied and implemented, they require an excessive amount of computational effort in order to represent a fairly simple result. The distribution function in the limit of ultimate distortion is simply a δ function in some direction. Similarly, the small-angle models are unable to accurately describe conditions of low anisotropy.

It would be very beneficial to find a simple representation of a distribution function that is similar to the two-term expansion for low anisotropy, but can distort into a δ function at large anisotropy. Such a representation would have all the desirable properties of the spherical harmonic expansion, but with proper asymptotic behavior at large anisotropy.

It will be shown that instead of expanding the distribution function in spherical harmonics, an ellipsoid of revolution can be used to represent f at any point of phase space. Analogous to the two-term spherical harmonic expansion, this ellipsoid can be described by two parameters (one scalar and one vector) that are functions of position, and the magnitude of velocity.

II. ASSUMED FORM OF THE DISTRIBUTION FUNCTION

Considering an ellipsoid of revolution, as shown in Fig. 1, the magnitude of the distribution function is taken to be the length of a line extending from one focus to a point on the surface. With one focus at the origin, it can be shown that

$$f=\frac{b\sqrt{1-\gamma^2}}{1-\vec{\gamma}\cdot\dfrac{\vec{v}}{v}},\tag{1}$$

where $\vec{\gamma}$ is a vector in the direction of the axis of symmetry, with magnitude equal to the eccentricity of the ellipsoid. By

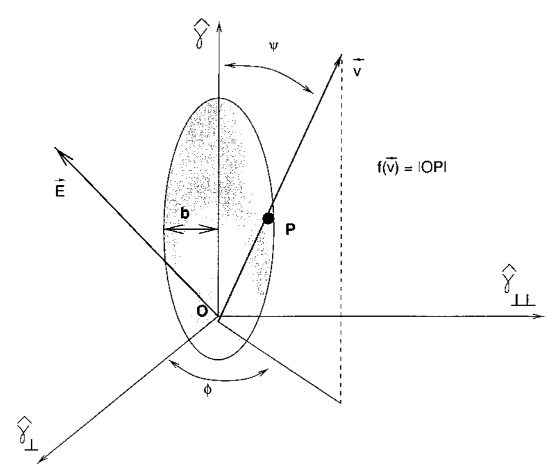

FIG. 1. The surface of an ellipsoid of revolution describes the distribution function in three dimensions. The electric field is shown for the general case in which it is not aligned with the axis of symmetry.

its definition, $0<\gamma<1$. Both b and $\vec{\gamma}$ are functions of position $\vec{r}$ and velocity magnitude v.

Equation (1) indicates that the elliptic representation is related to the two-term spherical harmonic expansion. For small values of γ, the fraction can be expanded in powers of γ. Retaining only first-order terms gives the approximation

$$f \approx b\left(1+\vec{\gamma}\cdot\frac{\vec{v}}{v}\right) \tag{2}$$

which is the two-term spherical harmonic expansion. For small γ, the relation between the two representations is given by

$$f_0(v)=b(v), \quad \vec{f}_1(v)=b(v)\vec{\gamma}(v). \tag{3}$$

All corrections to this by the elliptic representation are of second order, or higher, in γ. However, unlike the two-term spherical harmonic expansion, the elliptic representation is well behaved in the extreme cases for which $\gamma \Rightarrow 1$.

In some sense, Eq. (1) can be thought of as a generalization of the asymptotic approach of Long [8] as given in the Appendix of that work. However, unlike the present formulation, that analysis was not meant to have validity for low energy, or for nonconstant, nonisotropic collision cross sections. As a result of approximations used in the derivation of that result, the solution does not reduce to a solution of the two-term spherical harmonic expansion for low energies in the limit of low anisotropy. Furthermore, that analysis will not correctly represent the elastic regime of the distribution function, as the elastic in-scattering term is not accurately represented. This can be seen from the derivation of the anisotropy parameter in that work which has a lower limit of the order of the square root of the ratio of inelastic to elastic cross sections. As shown by Riemann [7] in the discussion surrounding his Eqs. (27)–(30), this parameter should scale as the square root of the ratio of energy loss to momentum loss frequencies. Thus, only in the inelastic regime will the analysis of Long have validity at low anisotropy, and, as can easily be shown, only then for energies much in excess of the mean energy gain between collisions divided by that same anisotropy parameter.

III. ANGULAR MOMENTS

With f given by Eq. (1), the Boltzmann equation can be integrated over all angles, so that two equations can be obtained to describe the evolution of b and $\vec{\gamma}$. In order to do this, it is convenient to define new quantities formed by integration over all solid angles:

$$n(\vec{r},v)=\int_\Omega f\,d\Omega=\frac{2\pi b\sqrt{1-\gamma^2}}{\gamma}\ln\left(\frac{1+\gamma}{1-\gamma}\right), \tag{4}$$

$$\vec{\Gamma}=\int_\Omega \frac{\vec{v}}{v} f\,d\Omega=\frac{2\pi b\sqrt{1-\gamma^2}}{\gamma}\left[\frac{1}{\gamma}\ln\left(\frac{1+\gamma}{1-\gamma}\right)-2\right]\hat{\gamma}, \tag{5}$$

and to define:

$$X=\frac{|\vec{\Gamma}|}{n}. \tag{6}$$

In general, the necessary integrals are performed by lining up the axes with $\vec{\gamma}$, a unit vector in the direction of $\vec{\gamma}$. The angles ϕ and ψ are then as shown in Fig. 1, and the general angular moment is obtained by the integral:

$$b\sqrt{1-\gamma^2}\int_{-1}^{+1}\int_0^{2\pi}\frac{F(\vec{v})}{1-\gamma x}\,d\phi\,dx, \tag{7}$$

where $x=\cos\psi$ and $F(\vec{v})$ is the function whose angular moment is to be taken. Obviously, $\vec{v}$ must be expressed in terms of v, ϕ, and x.

The correspondence to the two-term spherical harmonic expansion for small γ can be determined by looking at the first-order terms in Eqs. (4) and (5):

$$n \Rightarrow 4\pi b(v) \approx 4\pi f_0(v) \tag{8}$$

and

$$\vec{\Gamma} \Rightarrow \frac{4\pi}{3} b(v)\gamma\hat{\gamma} \approx \frac{4\pi}{3}\vec{f}_1, \tag{9}$$

$$X \Rightarrow \frac{\gamma}{3}. \tag{10}$$

Similarly, as $\gamma \rightarrow 1$, it follows that $X \rightarrow 1$ so that the magnitude of Γ becomes equal to n as the highly anisotropic δ function is approached.

Beginning with the Boltzmann equation in its usual form:

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\vec{\nabla}f-\frac{q_o\vec{E}}{m}\cdot\vec{\nabla}_v f=\left(\frac{\delta f}{\delta t}\right)_{coll}, \tag{11}$$

the first moment equation is obtained by integrating over solid angles Ω and the second is obtained by multiplying by $\vec{v}$, and then integrating over Ω. No assumption is made about the direction of the electric field with respect to the anisotropy. Thus, the general case in which $\hat{\gamma}\cdot\vec{E}\neq|\vec{E}|$ will be treated.

Some shortcuts will be useful in evaluating these various integrals. The first is to define axes perpendicular to $\hat{\gamma}$. The first such axis, $\hat{\gamma}_\perp$, is in the direction of $\delta\hat{\gamma}/\delta v$

$$\frac{\partial\hat{\gamma}}{\partial v}=\left|\frac{\partial\hat{\gamma}}{\partial v}\right|\hat{\gamma}_\perp, \tag{12}$$

while the second new axis, $\hat{\gamma}_{\perp\perp}$ is perpendicular to both:

$$\hat{\gamma}_{\perp\perp}=\hat{\gamma}\times\hat{\gamma}_\perp. \tag{13}$$

With these definitions, some simplifications are immediate:

$$\int_0^{2\pi}\vec{v}\,d\phi=2\pi v\cos\psi\hat{\gamma}, \tag{14}$$

$$\int_0^{2\pi}\vec{v}\vec{v}\,d\phi=\pi v^2(3\cos^2\psi-1)\hat{\gamma}\hat{\gamma}+\pi v^2\sin^2\psi\hat{\hat{I}}, \tag{15}$$

$$\int_0^{2\pi}\cos\phi\,\vec{v}\vec{v}\,d\phi=\pi v^2\cos\psi\sin\psi(\hat{\gamma}\hat{\gamma}_\perp+\hat{\gamma}_\perp\hat{\gamma}), \tag{16}$$

where

$$\hat{\hat{I}} = \hat{\gamma}\hat{\gamma} + \hat{\gamma}_\perp \hat{\gamma}_\perp + \hat{\gamma}_{\perp\perp} \hat{\gamma}_{\perp\perp} \tag{17}$$

is the identity tensor.

The various terms are as follows:

$$\int_\Omega \vec{v}\cdot\vec{\nabla} f d\Omega = \vec{\nabla}\cdot\int_\Omega f\vec{v} d\Omega = \vec{\nabla}\cdot(v\vec{\Gamma}), \tag{18}$$

$$\int_\Omega \frac{\vec{v}}{v}(\vec{v}\cdot\vec{\nabla} f) d\Omega = \vec{\nabla}\cdot\int_\Omega f\frac{\vec{v}\vec{v}}{v} d\Omega$$

$$= \vec{\nabla}\cdot\left[\frac{nv}{2\Gamma^2}\left(\frac{3X}{\gamma}-1\right)\vec{\Gamma}\vec{\Gamma}\right] + \vec{\nabla}\left[\frac{nv}{2}\left(1-\frac{X}{\gamma}\right)\right], \tag{19}$$

$$\int_\Omega \vec{E}\cdot\nabla_v f d\Omega = \frac{1}{v^2}\frac{\partial}{\partial v}(v^2\vec{E}\cdot\vec{\Gamma}), \tag{20}$$

$$\int_\Omega \frac{\vec{v}}{v}(\vec{E}\cdot\vec{\nabla}_v f) d\Omega = \int_\Omega \vec{E}\cdot(\vec{v}\vec{\nabla}_v f) d\Omega$$

$$= \frac{1}{v^3}\frac{\partial}{\partial v}\left[\frac{nv^3}{2\Gamma^2}\left(\frac{3X}{\gamma}-1\right)\vec{E}\cdot(\vec{\Gamma}\vec{\Gamma})\right] + \vec{E}\left\{\frac{\partial}{\partial v}\left[\frac{n}{2}\left(1-\frac{X}{\gamma}\right)\right] + \frac{n}{2v}\left(1-\frac{3X}{\gamma}\right)\right\}. \tag{21}$$

The moment equations are thus

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(v\vec{\Gamma}) - \frac{q_o}{m}\frac{1}{v^2}\frac{\partial}{\partial v}(v^2\vec{E}\cdot\vec{\Gamma}) = \left(\frac{\delta n}{\delta t}\right)_{coll}, \tag{22}$$

$$\frac{\partial\vec{\Gamma}}{\partial t} + \vec{\nabla}\cdot\left[\frac{nv}{2}\left(\frac{3X}{\gamma}-1\right)\hat{\gamma}\hat{\gamma}\right] + \vec{\nabla}\left[\frac{nv}{2}\left(1-\frac{X}{\gamma}\right)\right] - \frac{q_o}{m}\frac{1}{v^3}\frac{\partial}{\partial v}\left[\frac{nv^3}{2}\left(\frac{3X}{\gamma}-1\right)\vec{E}\cdot(\hat{\gamma}\hat{\gamma})\right] - \frac{q_o}{m}\vec{E}\left\{\frac{\partial}{\partial v}\left[\frac{n}{2}\left(1-\frac{X}{\gamma}\right)\right] + \frac{n}{2v}\left(1-\frac{3X}{\gamma}\right)\right\} = \left(\frac{\delta\vec{\Gamma}}{\delta t}\right)_{coll}. \tag{23}$$

It is interesting to note that in the small distortion limit, Eq. (10) forces some terms to vanish, and others to reach familiar limits. In particular,

$$\left(\frac{3X}{\gamma}-1\right) \Rightarrow 0 \tag{24}$$

and

$$\left(1-\frac{X}{\gamma}\right) \Rightarrow \frac{2}{3}. \tag{25}$$

These limits result in a replica (as they must) of the two-term spherical harmonic expansion.

As $\gamma\to 1$, the situation is different. In this case

$$\left(\frac{3X}{\gamma}-1\right) \Rightarrow 2 \tag{26}$$

and

$$\left(1-\frac{X}{\gamma}\right) \Rightarrow 0. \tag{27}$$

These limits serve to curtail further growth of $\vec{\Gamma}/n$ once it reaches the δ function limit of $v\hat{\gamma}$. The gradient terms, in both configuration space and velocity, disappear in Eq. (23) as $\gamma\to 1$. In this limit, the left-hand side of Eq. (22) becomes

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial z} - \frac{q_o E_z}{m}\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 n), \quad \gamma\to 1, \tag{28}$$

where z represents a coordinate in the direction of $\hat{\gamma}$. This is a purely advective operator. In fact, the left hand side operators of Eqs. (22) and (23) become proportional to each other, with a proportionality factor of v. As long as the collision terms do not permit further growth in the ratio $|\vec{\Gamma}|/n$, its value will be limited. This is as one would expect for a highly distorted ellipsoid. Equation (28) is the elliptic representation of the convection terms of a one-dimensional Boltzmann equation.

In order to use the elliptic representation, it is necessary to invert the following relation:

$$X = \frac{|\vec{\Gamma}|}{n} = \frac{1}{\gamma} - \frac{2}{\ln\left(\frac{1+\gamma}{1-\gamma}\right)}. \tag{29}$$

This is analogous to, and a generalization of, the equation given by Long [8] in that appendix, and again by Pitchford and Phelps [4] as their Eq. (27). The relation must be inverted quite often and with quite good accuracy in order to obtain the proper convergence of the various terms in the equations. Clearly, some approximation is needed. By matching the first, third, and fifth powers in the Taylor expansion, and requiring proper behavior at $X=1$ of both γ and $d\gamma/dX$, one can obtain the following polynomial ratio:

$$\frac{X}{\gamma} \approx \frac{35+92X^2+17X^4}{105+24X^2+15X^4}. \tag{30}$$

The function $X(\gamma)$, along with the ratio X/γ and its approximation, is shown in Fig. 2. Both X and γ are constrained to lie between -1 and 1.

IV. ELECTRON-HEAVY PARTICLE COLLISION TERMS

The general collision term for electrons with heavy particles is well known [9]. For an elastic interaction, this is written as

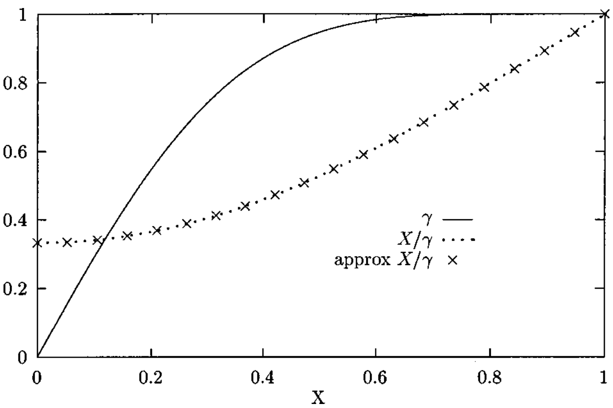

FIG. 2. The relation between γ and X, expressed in two different forms, along with the approximation of Eq. (30).

$$\left(\frac{\delta f}{\delta t}\right)_{c,el}=Nv\int_{\Omega'}\left[f(v'\mu')\left(\frac{v'}{v}\right)^4 q_{el}(\psi,v') - f(v,\mu)q_{el}(\psi,v)\right]d\Omega', \tag{31}$$

where $q_{el}(\phi,v)$ is the differential cross section for the collision with initial velocity v', final velocity v', and scattering angle ψ.

Following the technique of the previous section, angular moments can be taken of this collision term. Using the well-known geometric relations for scattering [9,10], and taking advantage of the azimuthal symmetry of the differential cross section and the distribution function from Eq. (1), one obtains

$$\left(\frac{\delta n}{\delta t}\right)_{c,el}=Nv\left(\frac{v'}{v}\right)^4\sigma_{el,T}(v')n(v')-Nv\sigma_{el,T}(v)n(v) \tag{32}$$

and

$$\left(\frac{\delta \vec{\Gamma}}{\delta t}\right)_{c,el}=Nv\left(\frac{v'}{v}\right)^4\sigma_{el,P}(v')\vec{\Gamma}(v')-Nv\sigma_{el,T}(v)\vec{\Gamma}(v), \tag{33}$$

where

$$\sigma_{el,T}(v)=2\pi\int_0^\pi q_{el}(\theta,v)\sin\theta d\theta \tag{34}$$

is the total collision cross section, and

$$\sigma_{el,P}(v)=\sigma_{el,T}(v)-\sigma_{el,M}(v) = 2\pi\int_0^\pi \cos\theta q_{el}(\theta,v)\sin\theta d\theta \tag{35}$$

has no common name, but can be described in terms of $\sigma_{el,M}(v)$, which is the momentum transfer cross section.

From Eqs. (32) and (33) it can be seen that each is composed of a repopulation term and a depopulation term. The relative depopulation rates are identical. However, owing to the presence of $\cos\theta$ in Eq. (35), it will always be true that $\sigma_{el,P}(v)<\sigma_{el,T}(v)$. As $|\vec{\Gamma}(v')|\rightarrow n(v')$ it will thus be guaranteed that the relative repopulation term for $\vec{\Gamma}$ will be less than that for n. This will ensure that the collisional processes will prevent $|\vec{\Gamma}|$ from growing to be greater than n, and is consistent with the intuitive notion that collisions should only broaden the angular extent of the distribution function. This property holds for the other types of collisions presented below. Thus, provided that collision terms are properly represented in the angular moment equations, collisional processes will serve to suppress further growth in X as it approaches unity.

For practical cases, approximations to the above terms can be made by the usual approach of taking the Taylor expansion of $f(v,\mu)$. This leads to the following relations, to first order in $\delta_{el}=2(m/M)$

$$\left(\frac{\delta n}{\delta t}\right)_{c,el}=N\frac{1}{2}\delta_{el}\frac{1}{v^2}\frac{\partial}{\partial v}(v^4\sigma_{el,M}(v)n) + N\frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial v}\left(\delta_{el}\sigma_{el,M}(v)v\frac{kT}{m}\frac{\partial n}{\partial v}\right) \tag{36}$$

and

$$\left(\frac{\delta \vec{\Gamma}}{\delta t}\right)_{c}=-Nv\sigma_{el,M}(v)\vec{\Gamma}+N\frac{1}{2}\delta_{el}\frac{1}{v^2}\frac{\partial}{\partial v}(v^4\sigma^1_{el,M}\vec{\Gamma}), \tag{37}$$

where

$$\sigma^1_{el,M}=2\pi\int_0^\pi(1-\cos\theta)q_{el}(\theta,v)\cos\theta\sin\theta d\theta \tag{38}$$

and the effects of atom recoil on $n(v)$ have been included. Although it is generally ignored, $\sigma^1_{el,M}$ should, strictly speaking, be taken into account. For collisions in which forward scattering dominates, $\sigma_{el,M}$ may become small enough that its effect on $|\vec{\Gamma}|$ relative to the effect of the derivative term on n may be questionable. The presence of the derivative term in Eq. (37), however, removes this concern, as $\sigma^1_{el,M}\rightarrow\sigma_{el,M}$ in such cases. Thus, aside from the small contribution of the atom recoil term, Eqs. (36) and (37) become similar and, in fact, proportional as q_{el} becomes more forward peaked and, hence, $X\rightarrow 1$. However, $\sigma_{el,M}$ is becoming so small in these cases that other terms will generally dominate. Furthermore, $\sigma^1_{el,M}$ is not generally available. The calculations of the next section will show that quite a range of conditions can be evaluated by ignoring $\sigma^1_{el,M}$ altogether.

For inelastic collisions, similar arguments apply. The result for an excitation collision is

$$\left(\frac{\delta n}{\delta t}\right)_{ex}=Nv\left(\frac{v'}{v}\right)^2\sigma_{ex,T}(v')n(v')-Nv\sigma_{ex,T}(v)n(v) \tag{39}$$

and

$$\left(\frac{\delta \vec{\Gamma}}{\delta t}\right)_{ex}=Nv\left(\frac{v'}{v}\right)^2\sigma_{ex,P}(v')\vec{\Gamma}(v')-Nv\sigma_{ex,T}(v)\vec{\Gamma}(v), \tag{40}$$

where

$$\sigma_{ex,T}(v)=2\pi\int_0^{\pi}q_{ex}(\theta,v)\sin\theta d\theta \qquad (41)$$

and

$$\sigma_{ex,P}(v)=2\pi\int_0^{\pi}q_{ex}(\theta,v)\cos\theta\sin\theta d\theta. \qquad (42)$$

Once again, the precise form of the collision terms is such that the diminishing effect on $|\vec{\Gamma}|$ is greater than on n. Further approximations can be made in order to simplify the equations, but these must be made carefully. For excitation energies that are large in comparison with a characteristic energy over which the distribution function changes greatly in magnitude, and for energies above the excitation energy, the depopulation terms will dominate, and it is appropriate to retain only those terms. At lower energies, the repopulation term can, of course, be significant in Eq. (39). However, if at the energies near threshold, q_{ex} is nearly isotropic, then $\sigma_{el,P}\approx 0$, and it is appropriate to ignore the repopulation term in Eq. (40). At energies much above the excitation threshold, and where characteristic energies are much greater than the excitation energy, all terms must be retained. For these energies, Altshuler [11] shows that a momentum transfer cross section can be defined as

$$\sigma_{ex,M}(v)=2\pi\int_0^{\pi}q_{ex}(\theta,v)(1-\cos\theta)\sin\theta d\theta \qquad (43)$$

so that for these cases

$$\left(\frac{\delta\vec{\Gamma}}{\delta t}\right)_{ex}=-Nv\sigma_{ex,M}(v)\vec{\Gamma}(v). \qquad (44)$$

In order to correctly account for the repopulation term where it is appropriate and to let it gracefully disappear at energies near threshold, Riemann [7] has derived a simple cross section based on the Thompson model for the inelastic momentum transfer cross section:

$$\sigma_{ex,M}=2\frac{uu_{ex}\ln\dfrac{u}{u_{ex}}}{u^2-u_{ex}^2}\sigma_{ex,T}(v), \qquad (45)$$

where $u=mv^2/2e$ is the electron energy, and u_{ex} is the excitation energy. This form, when used with Eqs. (44) and (39) is physically reasonable, and has been shown, computationally, to be adequate.

Ionization collisions are similar to excitation collisions, except for the creation of the new electron. Various approaches have been taken for the treatment of this electron. Margeneau [12] suggested that it share the resulting energy with the incident electron after the collision. Although this may be appropriate at low energies, it certainly is not at high energies where forward scattering dominates. The other common approach is to deposit all newly created electrons at energy zero, and to cope with the presence of a δ function in electron population there [13]. This latter approach will be used in the calculations of the next section. The ionization terms are then similar to excitation terms, and are

$$\left(\frac{\delta n}{\delta t}\right)_i=Nv\left(\frac{v'}{v}\right)^2\sigma_{i,T}(v')n(v')-Nv\sigma_{i,T}(v)n(v)$$
$$+N\frac{\omega_i}{4\pi v^2}\delta(0), \qquad (46)$$

$$\left(\frac{\delta\vec{\Gamma}}{\delta t}\right)_i=-Nv\sigma_{i,M}(v)\vec{\Gamma}(v), \qquad (47)$$

where

$$\omega_i=4\pi\int_0^{\infty}\sigma_{i,T}(v)vn(v)v^2dv, \qquad (48)$$

$$\sigma_{i,T}(v)=2\pi\int_0^{\pi}q_i(\theta,v)\sin\theta d\theta, \qquad (49)$$

and

$$\sigma_{i,M}=2\frac{uu_i\ln\dfrac{u}{u_i}}{u^2-u_i^2}\sigma_{i,T}(v), \qquad (50)$$

and u_i is the ionization energy.

It is important that whatever approximations are used to account for the collision processes, they must be chosen in a self-consistent manner. A good discussion of these considerations can be found in Phelps and Pitchford [5]. For example, any small-energy approximation for a momentum transfer cross section [Eq. (43)] must be applied only in situations where the excitation energy (or ionization energy) is small with respect to the characteristic energy. It has been shown above that since all terms representing a given collisional process are derived from the same differential cross section, the relative diminishing effect on Γ will be greater than that on n, as maximum anisotropy is approached. This property must be preserved in the approximations used, not to ensure the success of the elliptic representation, but in order to correctly represent the physics.

V. APPLICATION TO A TOWNSEND DISCHARGE

The Townsend discharge provides an example for the application of the elliptic representation to a simple situation. Following closely the analysis of Riemann [7], a stationary solution to the Boltzmann equation is sought in which all spatial dependency is assumed to vary as $e^{\alpha z}$, where z is the direction opposite the uniform applied field. With only one spatial dimension, $\vec{\Gamma}$ becomes a scalar quantity, and Eqs. (22) and (23) can be reduced to

$$\frac{\partial n}{\partial t}+\alpha v\Gamma+F\frac{1}{v^2}\frac{\partial}{\partial v}(v^2\Gamma)=\left(\frac{\delta n}{\delta t}\right)_{coll}, \qquad (51)$$

$$\frac{\partial\Gamma}{\partial t}+\frac{\alpha\Gamma v}{\gamma}+F\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\frac{n}{2}\left(\frac{3X}{\gamma}-1\right)\right]+F\frac{\partial}{\partial v}\left[\frac{n}{2}\left(1-\frac{X}{\gamma}\right)\right]=\left(\frac{\delta\Gamma}{\delta t}\right)_{coll}, \tag{52}$$

where $F=-eE/m$.

Several observations can be made at this time. From Eq. (51), it is immediately obvious that the Townsend coefficient α must satisfy the following relation in steady state:

$$\alpha=\frac{4\pi\int_0^\infty\left(\frac{\delta n}{\delta t}\right)_{coll}v^2dv}{4\pi\int_0^\infty\Gamma v^3dv}, \tag{53}$$

which is similar to Eq. (46) of Riemann [7]. Furthermore, in the high-energy region in which collisional losses can be neglected altogether, Eq. (51) gives us

$$\lim_{v\to\infty}\Gamma=\frac{A}{v^2}e^{-(\alpha/2F)v^2}, \tag{54}$$

where A is some arbitrary constant of proportionality. This is similar to Eq. (51) of [7]. It is also clear that, for sufficiently high energy, $\Gamma=n$, so that for sufficiently high $F/N,\alpha/F$ must tend to the limiting curve described by the following relation, derived from Eqs. (53) and (54):

$$\left(\frac{F}{N}\right)=\int_0^\infty\sigma_{i,T}(v)v e^{-(1/2)(\alpha/F)v^2}dv. \tag{55}$$

This is identical to Eq. (54) of [7]. However, it can be expected that use of the elliptic representation will result in convergence to this limiting curve gracefully, and without the application of the limiting step of Eq. (53) in [7]. That this is the case follows from the design of the elliptic representation in that it reduces to the equivalent one-dimensional description under extreme distortion, and should not be surprising in light of the discussion surrounding Eq. (55) of [7].

Some simplification can be obtained by changing the independent variable to total energy u and by defining $\eta=vn$ and $G=v^2\Gamma$. Also, by defining a ''energy velocity'' v_u and a ''energy pressure'' P_u as follows:

$$v_u=mFvX-Nu\delta_{el}\sigma_{el,M}(v)v, \tag{56}$$

$$P_u=2uF\left(\frac{X}{\gamma}\right)\eta, \tag{57}$$

the equations become very similar to the equations for time-dependent compressible gas flow. Using the approximations to the collisions terms as described in Sec. IV, this yields

$$\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial u}(v_u\eta)=N\frac{\partial}{\partial u}\left(u\delta_{el}\sigma_{el,M}vkT\frac{\partial\left(\frac{\eta}{v}\right)}{\partial u}\right)-\alpha G \tag{58}$$

$$+N(\eta'\sigma_{ex,T}(v')v'-\eta\sigma_{ex,T}(v)v+\eta'\sigma_{i,T}(v')v'-\eta\sigma_{i,T}(v)v+S_i\delta(0)) \tag{59}$$

and

$$\frac{\partial G}{\partial t}=-\frac{\partial P_u}{\partial u}-N[\sigma_{el,M}+\sigma_{ex,M}+\sigma_{i,M}]vG+\left(F-\alpha v^2\frac{X}{\gamma}\right)\eta \tag{60}$$

with

$$\alpha=N\frac{S_i}{\int_0^\infty G\,du} \tag{61}$$

and

$$S_i=\frac{1}{4\pi}\frac{m}{e}\omega_i=\int_0^\infty\eta(u)\sigma_{i,T}(u)v\,du. \tag{62}$$

With Eqs. (58) and (60), a solution for the distribution function in terms of η and G can be found by solving the time-dependent equations until the steady state is reached. These equations can be solved by the common techniques used in the study of time-dependent reactive flow. The left-hand sides contain convection operators [in Eq. (58)], and with zero velocity in Eq. (60), while the right-hand sides contain a diffusion term [in Eq. (58)], a pressure gradient term [in Eq. (60)], and numerous collisional (reaction) and body force terms. The solution scheme used here is based on the flux corrected transport method of Boris and Book [14], with fractional-step coupling to include the various source terms. The system is then advanced in time until the value of α has reached its asymptotic value.

Once convergence is obtained, it follows that the electron drift velocity can be obtained by evaluation of the following:

$$v_D=\frac{\int_0^\infty\Gamma v^3dv}{\int_0^\infty nv^2dv}=\frac{\int_0^\infty G\,du}{\int_0^\infty\eta\,du}. \tag{63}$$

Likewise, the average electron energy is determined by evaluation of

$$\overline{u}=\frac{\int_0^\infty nuv^2dv}{\int_0^\infty nv^2dv}=\frac{\int_0^\infty\eta u\,du}{\int_0^\infty\eta\,du}. \tag{64}$$

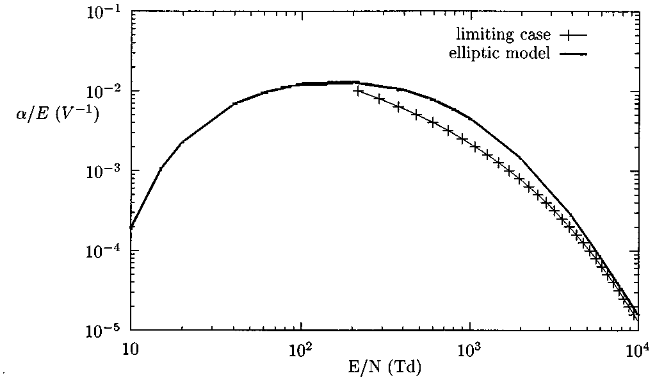

FIG. 3. Townsend ionization coefficient in helium showing detailed calculations and extreme result obtained from Eq. (55).

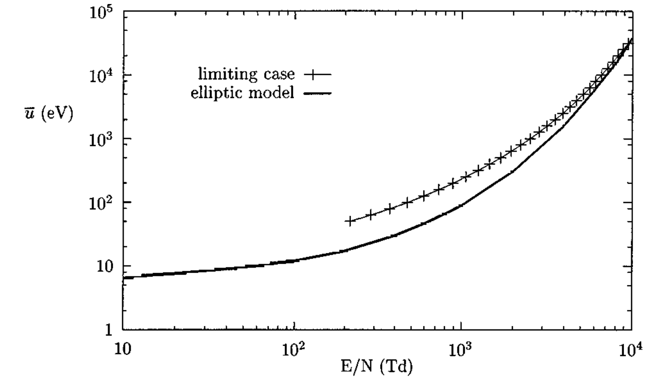

FIG. 5. Mean electron energy for a Townsend discharge in helium as found by the elliptic model, along with extreme result discussed in the text.

These equations have been integrated to steady-state conditions in helium using the same cross sections as used by Riemann [7]. These cross sections exhibit the runaway condition in which the momentum transfer cross section decreases faster than $1/u$. The results for these calculations are shown in Figs. 3–6. Figure 3 shows the Townsend coefficient, in the form of (α/E), along with the result of the limiting case of Eq. (55). The two curves are seen to come together at values of E/N which are very similar to that shown in Fig. 5 of [7]. This limit is approached smoothly without any artificial limiting of the components of the distribution function, even though cross sections with rapid decrease beyond the runaway condition were used.

Figure 4 shows the drift velocity at various values of E/N, from which it can be seen that a sufficiently large extent of E/N has been chosen that even the drift velocities are becoming relativistic. At the upper end of this range, the Boltzmann equation would require modification. Also shown is the limiting curve $v_D=\sqrt{(2/\pi)(F/\alpha)}$ derived from Eqs. (54) and (63) with $n=\Gamma$. Similarly, Fig. 5 shows the average energy of the distribution function as a function of E/N, along with the limiting curve $u_{av}=1/2(E/\alpha)$. As with the drift velocity, the energies in the range of high E/N are becoming quite large.

Figure 6 shows $X(u)$ for two values of E/N, showing how X grows with energy for sufficiently low energy values at which the Druyvesteyn-like behavior of the distribution function causes increase with energy of the asymmetric part of the distribution relative to the symmetric part. In terms of the classical two-term spherical harmonic expansion, this represents the inevitable growth with energy of the ratio $|\vec{f}_1/f_0|\approx|E\lambda(1/f_0)(df_0/du)|$ for any f_0 that has a tail falling faster than exponentially with energy. This growth is contained, however, by the effects of the elliptic representation. Hence, the rate of rise of $X(u)$ at 10 Td is already diminishing at 100 eV. At higher E/N, the bulk of the distribution function is highly directed, and $X\approx 1$ over nearly the entire range of relevant energies. These plots serve to show how the equations self-limit the growth of the ratio of anisotropic to isotropic components of the distribution, even in the presence of collision cross sections that fall off rapidly with energy.

VI. DISCUSSION

The similarity to the two-term spherical harmonic expansion must not be overlooked. In fact, by making the substitutions of $n=4\pi f_0$ and $\vec{\Gamma}=(4\pi/3)\vec{f}_1$, one obtains from Eqs. (22) and (23) exactly the two-term spherical harmonic expansion, provided that one makes the substitution

$$\vec{\vec{f}}_2=\frac{5}{4}\left(3\frac{X}{\gamma}-1\right)(3\hat{\gamma}\hat{\gamma}-\hat{\hat{I}})f_0. \tag{65}$$

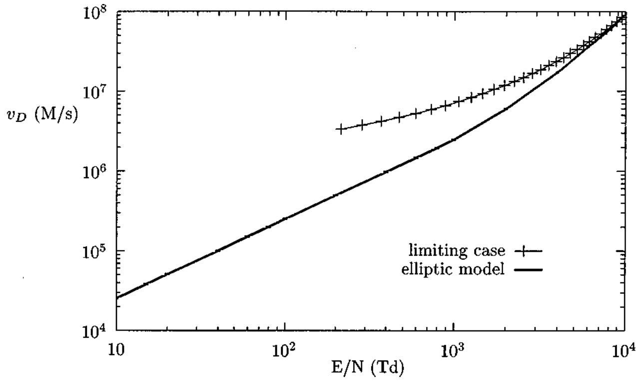

FIG. 4. Drift velocity in helium found by the elliptic representation, along with extreme result discussed in the text.

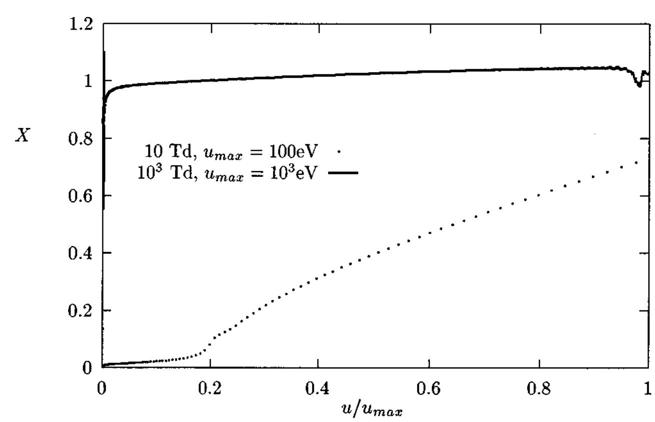

FIG. 6. Plots of $X=|\vec{\Gamma}|/n$ vs energy (normalized) for two values of E/N.

Thus, the elliptic representation can be thought of as being identically the two-term spherical harmonic expansion, with closure of the hierarchy according to Eq. (65).

$\vec{\vec{f}}_2$ as defined above is a valid and reasonable representation, as it satisfies several important requirements. It is a symmetric and traceless tensor, it must be (see, for example, Shkarofsky, Johnston, and Bachynski [3]). Furthermore, the ratio $\vec{\vec{f}}_2/f_0$ is second order in X. Riemann [7] explains that this must be the case. His analysis is in terms of the quantity $\rho = mv^2 N q_{el}/qE$, and applies to regions of phase space for which anisotropy is not great and, hence, $\rho \gg 1$. As $1/\rho \to 0$ we also have $X \to 0$, so that the order described by Riemann as

$$f_{n-1} = O(\rho f_n) \tag{66}$$

leads directly to the requirement that f_n must appear as $O(X^n)$ or, equivalently, as $O(\gamma^n)$. That the elliptic representation satisfies this requirement automatically becomes evident by performing a Taylor expansion of Eq. (1).

Although only the first three terms of the spherical harmonic expansion are ever actually used in the elliptic representation, the second-order nature of the closure of the hierarchy provides a realistic approximation. $\vec{\vec{f}}_2$ is highly suppressed when X is small, yet it reaches a magnitude of $5f_0$ as $X \to 1$, as it must in order to represent the unidirectional distribution.

Other approaches to this goal have been proposed. In particular, the maximum anisotropic approach of Baraff [15] is based on the spherical harmonic expansion, and terminates the hierarchy by choosing the ratio of the highest term to that of the next-to-highest term to be that corresponding to the description of a δ function. Thus, for two terms, this would yield $f_2 = (5/3)f_0$. Although this approach has been seen to yield good results for some transport coefficients, it still needs to include more than two terms in order to accurately represent the distribution function. This form of closure does not satisfy the order of magnitude requirement of Eq. (66), and might be expected to yield incorrect results for situations in which the bulk of the distribution has low anisotropy, especially when only a small number of terms is used. In particular, with only two terms, there would be present two quantities of first order in X, and one might expect the function f_1 so determined to be incorrect at low energies, as the terms in f_2 would be relatively significant. A possibly better approximate closure would be simply $f_2 = 5X^2 f_0 = 5/9(f_1^2/f_0)$, which is of second order and reaches the correct limit at unit eccentricity. A comparison of these various closures is shown in Fig. 7.

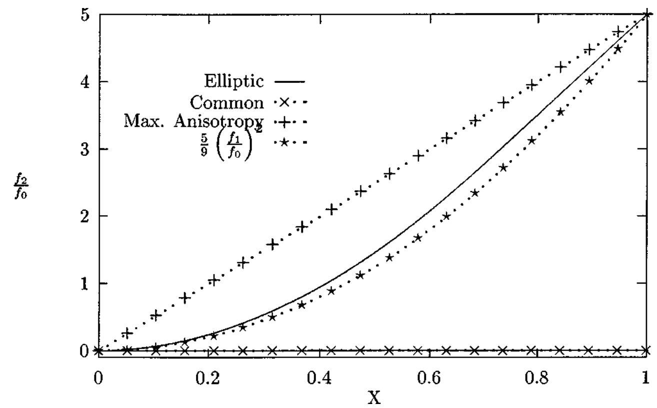

FIG. 7. Plot of the closure of the hierarchy of the two-term spherical harmonic expansion according to the usual closure ($f_2 = 0$), the maximum anisotropic closure of Baraff ($f_2 = (5/3)f_1$), and the elliptic representation [Eq. (65)]. Also shown is the approximation $5X^2$.

VII. SUMMARY

It has been shown that an alternative to the two-term spherical harmonic approximation can be derived that exhibits a smooth transition to a description of a highly distorted distribution function. This transition can be treated at the expense of only a slight increase in computational complexity. Furthermore, the framework of a traditional reactive flow transport description can be retained. It is expected that the analysis of situations which simultaneously exhibit both highly isotropic and highly anisotropic distributions will benefit from this approach.

An example calculation for a Townsend discharge in helium has been given. The results correspond to that of the two-term spherical harmonic expansion at low E/N, and to the theoretical limiting case derived by Riemann [7] at high E/N. The suppression of the u-gradient term in the anisotropic equation at high energies leads to a condition in which the isotropic and anisotropic portions are forced to follow the same equation. This leads to nearly unit eccentricity for such situations, showing that the formation of a beam is accurately represented. Using a time-dependent solution technique results were obtained over a very wide range of E/N, using realistic cross sections.

ACKNOWLEDGMENTS

The author would like to thank several people for valuable discussions and pointers, including J. H. Ingold, L. C. Pitchford, A. V. Phelps, Yan-Ming Li, and K.-U. Riemann.

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